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Truss structures with uncertain parameters - geometrical interpretation of the solution based on properties of convex sets

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Abstract

This paper addresses the problem of uncertain parameters in linear equations of truss structures. In general it is difficult to describe the appropriate fields within the space of displacements if uncertain parameters appears. The authors propose is to use the Minkowski sum of sets to describe appropriate displacements transferred to the space of loads. Instructive example is presented for a truss in 2D space of loads and displacements with three uncertain parameters.

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Keywords: Truss; Uncertain parameters; Interval analysis; Minkowski sum

1. Introduction

Uncertain coefficients that arise for uncertain geometry, material properties, or loads are common for engineering structures. The uncertain parameters are usually identified by random variables with stochastic approach or fuzzy set theory. The other possibility is to use the interval arithmetic [1,3] used with success for truss structures and finite element formulation [2,4]. The open problem, introduced in this paper, is an easy interpretation of the results, with special emphasis on geometrical interpretation. Linear equations are used for the analysis of truss structures.

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2. Formulation

The subject of the analysis is n-membered, supported truss with following characteristics of each member: Young modulus E_i, cross-sectional areas A_i and bar lengths L_i ($i=1,...,n$). Its mechanical properties are described by three linearized equations:
compatibility

$$\mathbf{\Delta} = \mathbf{Bq}\,, \tag{1}$$

material properties

$$\mathbf{S} = \mathbf{D\Delta}, \tag{2}$$

and equilibrium

$$\mathbf{B}^T\mathbf{S} = \mathbf{P}, \tag{3}$$

with boundary conditions included, where $\mathbf{q}$ is the displacement vector of length k, $\mathbf{\Delta}$ the extension vector of length n, $\mathbf{S}$ the normal force vector of length n, $\mathbf{D}$ the elasticity matrix, $\mathbf{P}$ the load vector and $\mathbf{B}$ the compatibility matrix which can be determined directly or using the formalism of the finite element method [5].

Substituting (1) to (2) and to (3) one can obtain a linear system of algebraic equations

$$\mathbf{Kq} = \mathbf{P} \tag{4}$$

where $\mathrm{K} \in \mathbb{R}^{n\times n}$; $\mathrm{q}, \mathrm{P} \in \mathbb{R}^{n}$, $\mathrm{D} \in \mathbb{R}^{k\times k}$ – diagonal matrix of parameters $\mathrm{D_i}$ $(\mathrm{i} = 1, \dots, \mathrm{k})$.

Let's define intervals $\mathrm{D_i} \in [\mathrm{D_i}]$, $[\mathrm{D}]$ – interval diagonal matrix with the elements $[\mathrm{D_i}]$ $(\mathrm{i} = 1, \dots, \mathrm{k})$ on the diagonal; $\mathrm{K} = \mathrm{B^T D B}$; $\mathrm{B} \in \mathbb{R}^{k\times n}$; $\mathrm{B^T} \in \mathbb{R}^{n\times k}$; $\mathrm{B^{(i)}}$- i-th column of the matrix $\mathrm{B^T}$ $(\mathrm{i} = 1, \dots, \mathrm{k})$; $\mathrm{B^{(i)}} \in \mathbb{R}^{n}$.

The system of equations (4) can be expressed in the form

$$\mathbf{Kq} = \mathbf{P} \iff \textstyle\sum_{i=1}^{k} \langle \mathbf{B}^{(i)}, \mathbf{q} \rangle\, D_i \mathbf{B}^{(i)} = \mathbf{P}, \tag{5}$$

where $\langle \mathrm{B^{(i)}}, \mathrm{q} \rangle$ is a scalar product of vectors $\mathrm{B^{(i)}}$ and q.

The solution set can be expressed in the form

$$\Xi = \{\mathbf{q} \in \mathbb{R}^n | (\exists \mathbf{D} \in [\mathbf{D}])((\mathbf{B}^T\mathbf{D}\mathbf{B})\mathbf{q} = \mathbf{P})\} \tag{6}$$

with the family of linear algebraic systems

$$\mathbf{Kq} = \mathbf{P}, \mathbf{K} \in [\mathbf{K}] = \mathbf{B}^T[\mathbf{D}]\mathbf{B} = \{\mathbf{B}^T\mathbf{D}\mathbf{B} | \mathbf{D} \in [\mathbf{D}]\} \tag{7}$$

$$[\boldsymbol{K}]\mathbf{q} = \mathbf{P}. \tag{8}$$

The family of systams (7) can be written symbolically in the form

$$\mathbf{q} \in \Xi \Leftrightarrow \mathbf{P} \in \boldsymbol{\mathcal{V}}(\mathbf{q}),\ \boldsymbol{\mathcal{V}}(\mathbf{q}) = \textstyle\sum_{i=1}^{k} \langle \mathbf{B}^{(i)}, \mathbf{q} \rangle\, \mathbf{B}^{(i)} * [D_i], \tag{9}$$

Where „$*$" stands for the membervise multiplication of sets. A set $\mathcal{V}(\mathrm{q})$ is the Minkowski sum of k line segments in the $\mathbb{R}^n$ space. The line segments are parallel to the vectors $\mathrm{B}^{(1)}, \dots, \mathrm{B}^{(k)}$. In general it is difficult to describe the area of acceptable displacements q. The idea of the present paper is to move the description of acceptable q space of loads P described via a Minkovski sum in the formulae (9)

3. Truss example

To present a geometric interpretation of the solution let as consider a simple truss example with $\boldsymbol{q} \in \mathbb{R}^2$ (Fig.1). The matrices **B** and **D** are expressed by (10). The stiffness matrix **K** can be expressed in the form (11) with the use of uncertain parameters **s** and precisely defined load vector **P**. Left hand side of the equation is (8) with uncertain parameters $s_i (i = 1,2,3)$ that belongs to the intervals $s_i \in \mathbf{s}_i$ is expressed by (13). One can define a set of all possible solutions **q** of the problem as a united solution set (6).

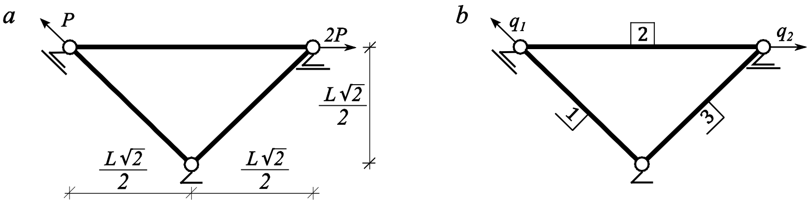

Fig. 1. Truss structure: geometry, load (a) and nodal displacements (b).

$$\mathbf{B} = \begin{bmatrix} 1 & 0 \\ \frac{\sqrt{2}}{2} & 1 \\ 0 & \frac{\sqrt{2}}{2} \end{bmatrix}, \quad \mathbf{D} = \begin{bmatrix} \frac{E_1 A_1}{L} & 0 & 0 \\ 0 & \frac{E_2 A_2}{L\sqrt{2}} & 0 \\ 0 & 0 & \frac{E_3 A_3}{L} \end{bmatrix}. \tag{10}$$

$$\mathbf{K}(\mathbf{s}) = \mathbf{K}(s_1, s_2, s_3,) = s_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + s_2 \begin{bmatrix} \frac{1}{2\sqrt{2}} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{\sqrt{2}} \end{bmatrix} + s_3 \begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}, \tag{11}$$

$$\mathbf{s} = (s_1, s_2, s_3,), \quad \mathbf{P} = \begin{bmatrix} P \\ 2P \end{bmatrix}, \quad s_i = \frac{E_i A_i}{L_i}, \tag{12}$$

$$\mathbf{K}(s_1, s_2, s_3,)\mathbf{q} = s_1 q_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + s_2 \left(\frac{1}{\sqrt{2}} q_1 + q_2 \right) \frac{1}{\sqrt{2}} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 1 \end{bmatrix} + s_3 q_3 \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix}. \tag{13}$$

It is possible to define a set

$$S(\mathbf{q}) = \{\mathbf{K}(s)\mathbf{q} \,|\, s \in \mathbf{s}\}, \tag{14}$$

which is the Minkowski sum of three line segments.

For the example the Minkowski sum is the area inside a hexagon with the sides parallel to the vectors

$\begin{bmatrix}1\\0\end{bmatrix}$, $\begin{bmatrix}\frac{1}{\sqrt{2}}\\1\end{bmatrix}$, $\begin{bmatrix}0\\\frac{1}{\sqrt{2}}\end{bmatrix}$, taken as the columns of the matrix $\mathbf{B}^T$. If trusses are considered the sides of Minkowski sum are straight. Position of the hexagon in the two dimensional space of loads depends on the matrix $\mathbf{K}(s)$, uncertainity of the parameters (intervals $\mathbf{s}_i$) and displacements $\mathbf{q}$. It does not depends on the load vector $\mathbf{P}$.

The length of sides depends on the scalar product $\langle \boldsymbol{B}^{(i)}, \boldsymbol{q} \rangle$ multiplied by the intervals of parameters (linearly depending on the intervals $\mathbf{s}_i$).

The calculations below are presented for $\frac{EA}{L} = 354$ with uncertainities on the level $\pm 2\%$ for each member of the truss and for load multiplier $P=1$. Then $\mathbf{s}_i = \left[354 - 7.08, 354 + 7.08\right]$ $(i = 1,2,3)$.

Properties of the set of acceptable displacements are the following

$$\boldsymbol{q} \in \boldsymbol{\varXi} \Leftrightarrow \boldsymbol{P} \in S(\boldsymbol{q}). \tag{15}$$

Minkowski sums for three displacement vectors $\mathbf{q}_1 = \begin{bmatrix}0.000423\\0.004505\end{bmatrix}, \mathbf{q}_2 = \begin{bmatrix}0.000438\\0.004570\end{bmatrix}, \mathbf{q}_1 = \begin{bmatrix}0.000440\\0.004620\end{bmatrix}$ are presented in Figure 2. Load vector $\mathbf{P}$ is expressed by a dot.

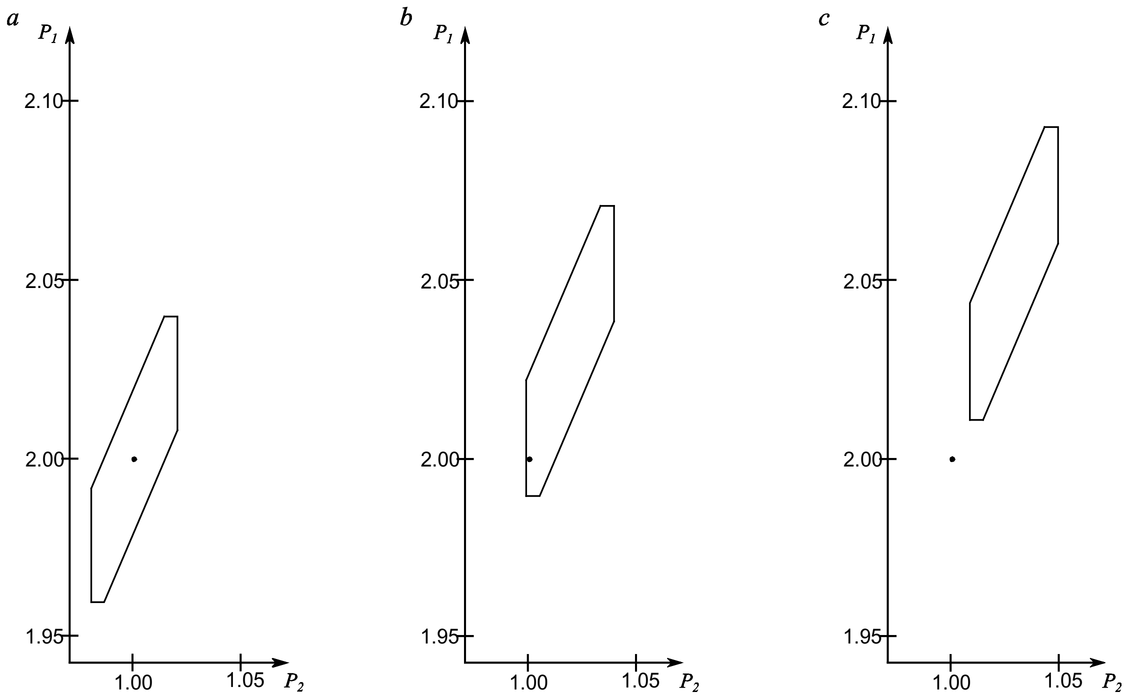

Fig. 2. Minkowski sums for displacement vectors $\mathbf{q}_1$ (a), $\mathbf{q}_2$ (b) and $\mathbf{q}_3$ (c).

Displacements $\mathbf{q}_1$ are acceptable, $\mathbf{q}_2$ are still acceptable and $\mathbf{q}_3$ are unacceptable.

4. Conclusions

The problem of uncertain parameters in linear equations of truss structures was introduced and discussed. Since it is difficult to describe the appropriate fields within the space of displacements if uncertain parameters appears the authors propose to describe the Minkowski sum as a space of appropriate displacements transferred to the set of loads. Instructive example is presented for a truss in 2D space of loads and displacements with three uncertain parameters. The results can be easily extended to N-dimensional space.

References

[1] G. Alefeld, G. Mayer, Interval analysis: theory and applications. J. of Computational and Applied Mathematics, 121, pp. 421-464, 2000.
[2] R.L. Muhanna, R.L. Mullen, H. Zhong, Penalty-based solution for interval finite-element methods. J. of Engineering Mechanics, pp. 1102-1111, October 2005.
[3] A. Neumaier, Introduction to numerical analysis. Cambridge University Press, Cambridge 2001.
[4] A. Neumaier, A. Pownuk, Linear systems with large uncertainities, with applications to truss structures. Reliable Comp., 13, pp.149-172,2007.
[5] O.C. Zienkiewicz, R.L. Taylor, The finite element method. Vol. 1. The basis. Butterworth, Heinemann, London 2000.